Transient open channel flow

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SWARM Summer School
15 – 26 November 2021

17th November 2021

Outline I

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Implicit Euler scheme

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Example:

$$u' = 4tu \qquad u(0) = 2 \tag{1}$$

Analytical solution:

$$u(t) = 2e^{2t^2} \tag{2}$$

Implicit time discretization of equation (1):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u' = 4tu \Rightarrow \frac{\Delta u}{\Delta t} = 4tu \Rightarrow \frac{u_{n+1} - u_n}{\underbrace{t_{n+1} - t_n}_{h}} = 4t_{n+1} \cdot u(t_{n+1}) \tag{3}$$

$$\Rightarrow u_{n+1} = u_n + 4 \cdot h \cdot t_{n+1} \cdot u_{n+1}$$

$$\Rightarrow u_{n+1} = \frac{u_n}{1 - 4ht_{n+1}} \tag{4}$$

Implicit Euler scheme

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	A	B
1	u0	2
2	h	0.1

(a) input

	A	B	C	D
1	i	t	u implicit Euler	u analytical
2	0	0	2.0000	2.0000
3	1	0.1	2.0833	2.0404
4	2	0.2	2.2645	2.1666
5	3	0.3	2.5733	2.3944
6	4	0.4	3.0634	2.7543
7	5	0.5	3.8293	3.2974
8	6	0.6	5.0385	4.1089
9	7	0.7	6.9980	5.3289
10	8	0.8	10.2912	7.1933
11	9	0.9	16.0799	10.1062
12	10	1	26.7999	14.7781

(b) solution

Fig.: Ordinary differential equation solved in Excel using implicit time discretization

Implicit Euler scheme

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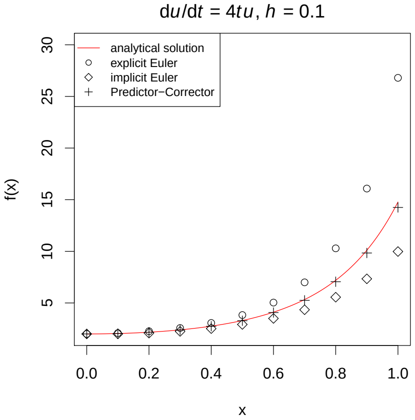
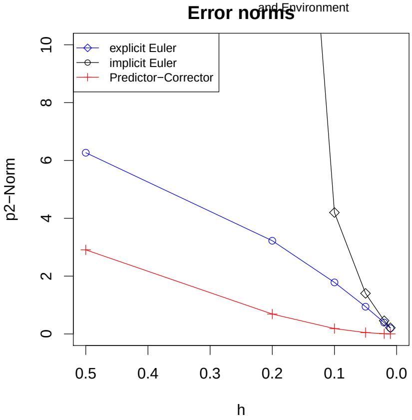


(a) solution



(b) $p2$-norms

Partial differential equaitons

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Example: Advecdtion-equation (see also mass transport):

$$\frac{\partial \phi}{\partial t} + u \cdot \frac{\partial \phi}{\partial x} = 0 \tag{5}$$

Initial condition:

$$\phi(x, 0) = \begin{cases} 1 & \text{für } 0 < x \leq 1 \\ 0 & \text{sonst} \end{cases} \tag{6}$$

to solve on $x = [0, 20]$ with boundary conditions:

$$\phi(0, t) = f(20, t) = 0 \tag{7}$$

Partial differential equaitons: Explicit time discretization

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Forward in time and backward in space (upwind):

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u \cdot \frac{\phi_i^n - \phi_{i-1}^n}{\Delta x} = 0 \Rightarrow \phi_i^{n+1} = \phi_i^n - \Delta t \cdot u \cdot \frac{\phi_i^n - \phi_{i-1}^n}{\Delta x} \quad (8)$$

Forward time, central space (FTCS):

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u \cdot \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2 \cdot \Delta x} = 0 \Rightarrow \phi_i^{n+1} = \phi_i^n - \Delta t \cdot u \cdot \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2 \cdot \Delta x} \quad (9)$$

Partial differential equaitons: Explicit time discretization

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	A	B
1	u	1
2	Delta t	0.5
3	Delta x	1
4	CFL R	0.5

(c) input

	A	B	C	D	E	F	G
1	n/i		0	1	2	3	4
2		t/x	0	1.000	2.000	3.000	4.000
3	0	0.0	0	1.000	0.000	0.000	0.000
4	1	0.5	0	0.500	0.500	0.000	0.000
5	2	1.0	0	0.250	0.500	0.250	0.000
6	3	1.5	0	0.125	0.375	0.375	0.125
7	4	2.0	0	0.063	0.250	0.375	0.250
8	5	2.5	0	0.031	0.156	0.313	0.313
9	6	3.0	0	0.016	0.094	0.234	0.313
10	7	3.5	0	0.008	0.055	0.164	0.273
11	8	4.0	0	0.004	0.031	0.109	0.219
12	9	4.5	0	0.002	0.018	0.070	0.164
13	10	5.0	0	0.001	0.010	0.044	0.117
14	11	5.5	0	0.000	0.005	0.027	0.081
15	12	6.0	0	0.000	0.003	0.016	0.054
16	13	6.5	0	0.000	0.002	0.010	0.035
17	14	7.0	0	0.000	0.001	0.006	0.022
18	15	7.5	0	0.000	0.000	0.003	0.014

(d) upwind scheme

Fig.: Solution of the advection equation (5) with $\Delta t = 0{,}5$ in Excel

Partial differential equaitons: Explicit time discretization

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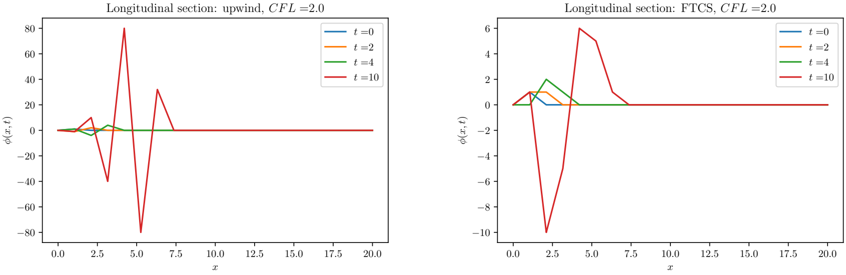

(a) upwind scheme

(b) FTCS scheme

Fig.: Solution of the advection equation (5) with $\Delta t = 2{,}0$

Partial differential equaitons: Explicit time discretization

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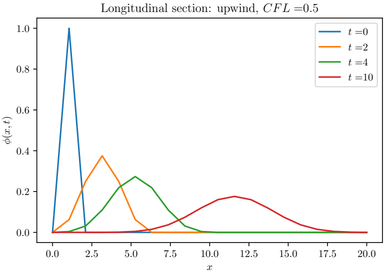
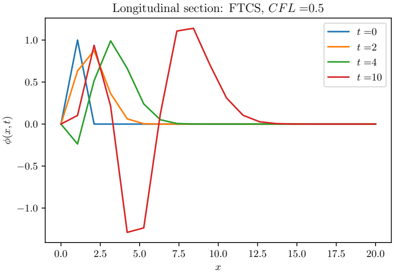


(a) upwind scheme



(b) FTCS scheme

Fig.: Solution of the advection equation (5) with $\Delta t = 0{,}5$

Partial differential equaitons: Explicit time discretization

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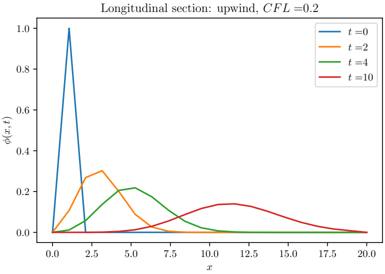
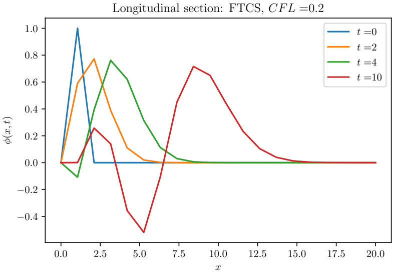


(a) upwind scheme



(b) FTCS scheme

Fig.: Solution of the advection equation (5) with $\Delta t = 0{,}2$

Partial differential equaitons: Stability and CFL Criterion

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For numerical stability of schemes using explicit time discretization the Courant number (CFL or R) needs to fall below a certain value, usually < 1:

$$CFL = R = \frac{\Delta t}{\Delta x} \cdot \sup_i |f'(u_i)| \tag{10}$$

Simplification of equation (10) for the one-dimensional advection equation with constant u:

$$CFL = R = \frac{\Delta t}{\Delta x} \cdot u \tag{11}$$

Partial differential equaitons: Stability and CFL Criterion

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Example: Estimate the Courant number for the numerical solution of an advection equation given the following values!

- $u = 2\,\mathrm{m\,s^{-1}}$
- $\Delta x = 3\,\mathrm{m}$
- $\Delta t = 2\,\mathrm{s}$

Equation (11):

$$CFL = \frac{\Delta t}{\Delta x} \cdot u = 1{,}3$$

required time step for $CFL < 1$:

$$\rightarrow \Delta t < CFL \cdot \frac{\Delta x}{u} \Rightarrow \Delta t < 1{,}5\,\mathrm{s}$$

Partial differential equaitons: Flood wave

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A flood wave is flowing through the following river section. In addition to the given values an initial condition $h(x, t = 0)$ and boundary condition $h(0, t)$ need to be estimated

- length L of the river section: 140 km
- width B of the river section: 30 m
- bed slope I: 0,001
- roughness k_{St}: $40\,\mathrm{m}^{1/3}\,\mathrm{s}^{-1}$ ($C = k_{\mathrm{St}} \cdot R^{\frac{1}{6}}$)
- constant flow rate: $560\,\mathrm{m}^3\,\mathrm{s}^{-1}$

Continuous gauge data is available from the start of the section

- maximum water depth of 7,96 m is observed after 6 hours
- after 24 h the constant flow rate of $560\,\mathrm{m}^3\,\mathrm{s}^{-1}$ is reached again

Partial differential equaitons: Flood wave

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Tasks `Hochwasserwelle_expl.xlsm`

1. What is the value of the peak discharge at the upper end the river section
2. At what time does this peak discharge occur in the middle of the river section
3. What is the value of the peak discharge in the middle of the river section?
4. What is the range of R (stability criterion CFL)?

Partial differential equaitons: Implicit time discretization

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Forward time and backward space discretization (implicit upwind):

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + u \cdot \frac{\phi_i^{n+1} - \phi_{i-1}^{n+1}}{\Delta x} = 0 \tag{12}$$

$$\Rightarrow \phi_i^{n+1} + \underbrace{u \cdot \frac{\Delta t}{\Delta x}}_{c} \cdot \left(\phi_i^{n+1} - \phi_{i-1}^{n+1}\right) = \phi_i^n$$

$$-c \cdot \phi_{i-1}^{n+1} + (1 + c) \cdot \phi_i^{n+1} = \phi_i^n \tag{13}$$

Partial differential equaitons: Implicit time discretization

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Using boundary conditions ϕ_0 and ϕ_{N_x} in equation (13) the following system of equations is obtained:

$$\begin{pmatrix} 1+c & 0 & 0 & 0 & & \dots & & 0 \\ -c & 1+c & 0 & 0 & & \dots & & 0 \\ 0 & -c & 1+c & 0 & & \dots & & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & \dots & & 0 & -c & 1+c & 0 & 0 \\ 0 & \dots & & 0 & 0 & -c & 1+c & 0 \\ 0 & \dots & & 0 & 0 & 0 & -c & 1+c \end{pmatrix} \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N_x-3} \\ \phi_{N_x-2} \\ \phi_{N_x-1} \end{pmatrix}^{n+1} = \begin{pmatrix} \phi_1 + c\phi_0 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{N_x-3} \\ \phi_{N_x-2} \\ \phi_{N_x-1} \end{pmatrix}^{n} \qquad (14)$$

Partial differential equaitons: Implicit time discretization

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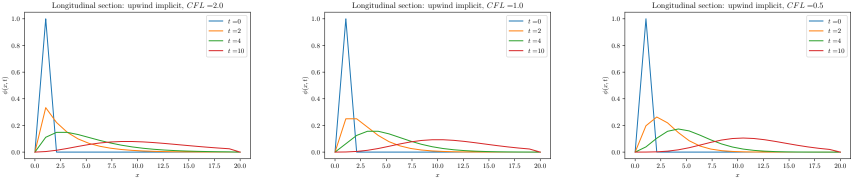

(a) $\Delta t = 2,0$ (b) $\Delta t = 1,0$ (c) $\Delta t = 0,5$

Fig.: Solution of the advection equation (5) using implicit time discretization (implicit upwind)

Many schemes using implicit time discretization are stable even for $CFL > 1$

Partial differential equaitons: Convergence

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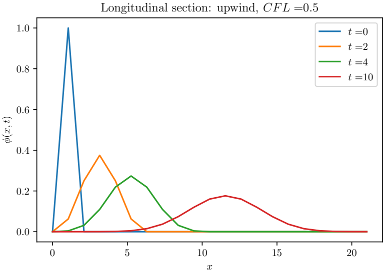
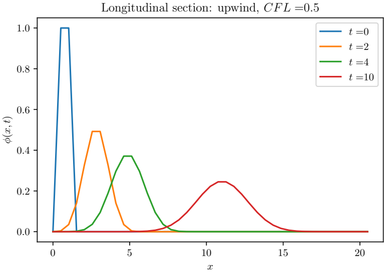


(a) $\Delta x = 1,00$



(b) $\Delta x = 0.50$

Partial differential equaitons: Convergence

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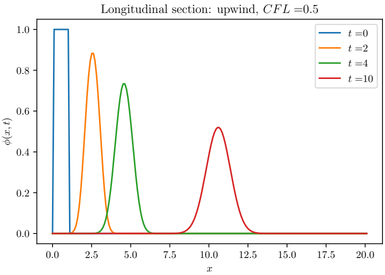
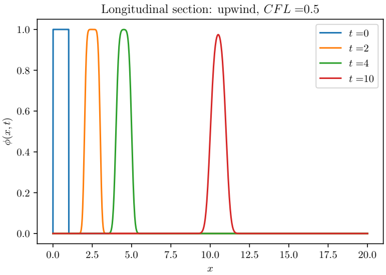


(c) $\Delta x = 0,10$



(d) $\Delta x = 0,01$

Timestep for $\Delta x = 0{,}01$ and $CFL = 0{,}5$:

$$\Delta t = \frac{\Delta x}{u} \cdot 0{,}5 = 0{,}005 \rightarrow 4\,000 \text{ timesteps}, 2\,000 \text{ points in space}$$

Partial differential equaitons: Quality of a numerical model

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- Consistency: $\Delta x \rightarrow 0$, $\Delta t \rightarrow 0$: discretization error $\rightarrow 0$
- Stability: error bounded for the entire flow field
- Convergence = Consistency + Stability (Lax' theorem)

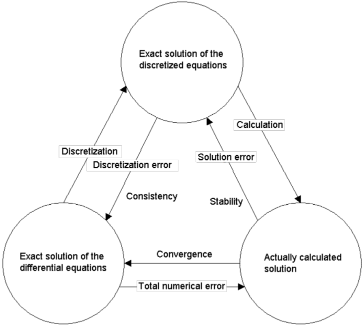

Fig.: Quality of a numerical model (source: M. Tritthart)

Transient open channel flow: Governing equations and discretization

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Continuity (rectangular cross section):

$$b \cdot \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (15)$$

Equation of motion (momentum):

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \cdot \left(\beta \cdot \frac{Q^2}{A} \right) + g \cdot A \cdot \left(\frac{\partial h}{\partial x} - I_0 + I_R \right) = 0 \quad (16)$$

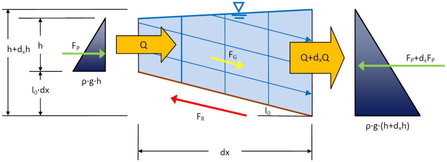

Fig.: Equation of motion (16) (Aubrunner, 2009)

Transient open channel flow: Governing equations and discretization

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Discreitization of the continuety equation (15) using the Euler scheme or the Predictor-Corrector scheme:

$$b \cdot \frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{Q_{i+1}^n - Q_{i-1}^n}{2 \cdot \Delta x} = 0 \qquad (17)$$

Transient open channel flow: Governing equations and discretization

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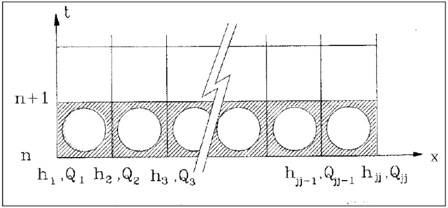
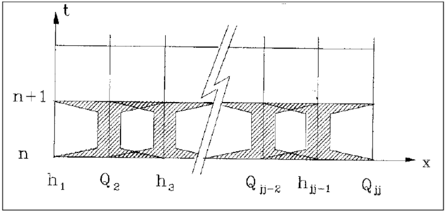
Discretization of the equation of motion (16) using:

Preissmann-Scheme: h and Q unknown at all mesh points

Abbott-Inesco Schema: unknown properties are defined alternately on the mesh (staggered-grid scheme)



(a) Preissmann-Scheme



(b) Abbott-Inesco Scheme

Transient open channel flow: Flood wave with an implicit scheme

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Study the flood wave from above using an implicit scheme

Tasks `Hochwasserwelle_impl.xlsm`

1. What is the value of the peak discharge at the upper end the river section
2. At what time does this peak discharge occur in the middle of the river section
3. What is the value of the peak discharge in the middle of the river section?
4. What is the range of R (stability criterion CFL)?

Summary

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Stencil of an explicit and an implicit scheme:

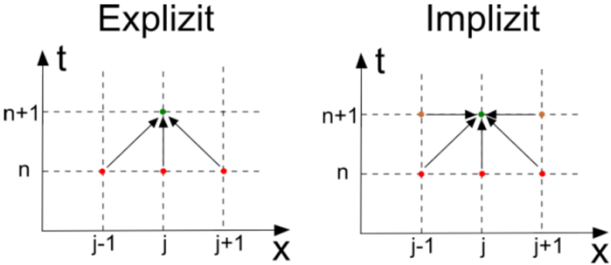

Fig.: Comparison of an explicit and and implicit scheme

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Literatur I

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Aubrunner, B. (2009). ‘Ein arbeitsblattbasiertes Programm zur eindimensionalen Berechnung des instationären Abflusses in offenen Gerinnen’. Masterarbeit. Universität für Bodenkultur Wien.