University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Levelling of connected tanks and mass transport

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Outline I

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Finite Differences Method (FDM): Derivative of a polynomial

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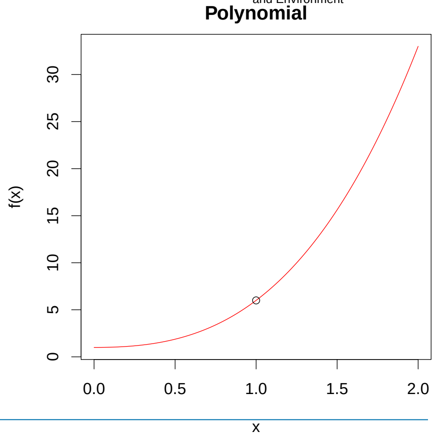
$$f(x) = 3x^3 + 2x^2 + 1 \qquad (1)$$

to be evaluated: slope at $x = 1$

$$\rightarrow f'(x) = 9x^2 + 4x \qquad (2)$$

solution from (2): $f'(1) = 13$



FDM: Definition of the slope from the differential

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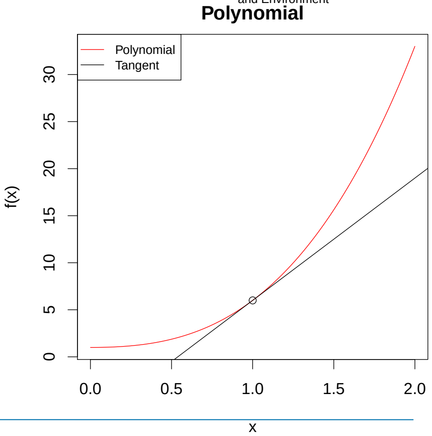
$$\frac{\partial f}{\partial x}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (3)$$

$$\frac{\partial f}{\partial x}(x) = \lim_{\Delta x \to 0} \frac{f(x) - f(x-\Delta x)}{\Delta x} \qquad (4)$$

$$\frac{\partial f}{\partial x}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x-\Delta x)}{2 \cdot \Delta x} = \frac{1}{2} \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x-\Delta x)}{\Delta x} \qquad (5)$$



FDM: Approximation of the slope from a finite difference

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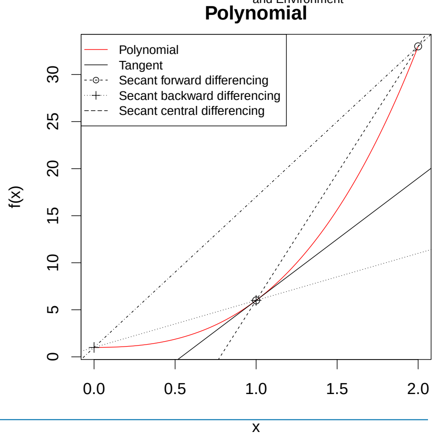
$$\frac{\partial f}{\partial x}(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{6}$$

$$\frac{\partial f}{\partial x}(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x} \tag{7}$$

$$\frac{\partial f}{\partial x}(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2 \cdot \Delta x} \tag{8}$$



FDM: Approximation of the derivative in Excel

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	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	x	1													
2	f(x)	6													
3	Δx	x+Δx	x-Δx	f(x+Δx)	f(x-Δx)	forward differencing	backward differencing	central differencing	f'(x)	Difference forward differencing	Difference backward differencing	Diffemce central differencing	error forward differencing	error backward differencing	error central differencing
4	2.000	3.000	-1.000	100.000	0.000	47.000	3.000	25.000	13.000	34.000	-10.000	12.000	34.000	10.000	12.000
5	1.000	2.000	0.000	33.000	1.000	27.000	5.000	16.000	13.000	14.000	-8.000	3.000	14.000	8.000	3.000
6	0.900	1.900	0.100	28.797	1.023	25.330	5.530	15.430	13.000	12.330	-7.470	2.430	12.330	7.470	2.430
7	0.800	1.800	0.200	24.976	1.104	23.720	6.120	14.920	13.000	10.720	-6.880	1.920	10.720	6.880	1.920
8	0.700	1.700	0.300	21.519	1.261	22.170	6.770	14.470	13.000	9.170	-6.230	1.470	9.170	6.230	1.470
9	0.600	1.600	0.400	18.408	1.512	20.680	7.480	14.080	13.000	7.680	-5.520	1.080	7.680	5.520	1.080
10	0.500	1.500	0.500	15.625	1.875	19.250	8.250	13.750	13.000	6.250	-4.750	0.750	6.250	4.750	0.750
11	0.400	1.400	0.600	13.152	2.368	17.880	9.080	13.480	13.000	4.880	-3.920	0.480	4.880	3.920	0.480
12	0.300	1.300	0.700	10.971	3.009	16.570	9.970	13.270	13.000	3.570	-3.030	0.270	3.570	3.030	0.270
13	0.200	1.200	0.800	9.064	3.816	15.320	10.920	13.120	13.000	2.320	-2.080	0.120	2.320	2.080	0.120
14	0.100	1.100	0.900	7.413	4.807	14.130	11.930	13.030	13.000	1.130	-1.070	0.030	1.130	1.070	0.030
15	0.050	1.050	0.950	6.678	5.377	13.558	12.458	13.008	13.000	0.558	-0.543	0.008	0.558	0.543	0.008
16	0.020	1.020	0.980	6.264	5.744	13.221	12.781	13.001	13.000	0.221	-0.219	0.001	0.221	0.219	0.001
17	0.010	1.010	0.990	6.131	5.871	13.110	12.890	13.000	13.000	0.110	-0.110	0.000	0.110	0.110	0.000
18	0.001	1.001	0.999	6.013	5.987	13.011	12.989	13.000	13.000	0.011	-0.011	0.000	0.011	0.011	0.000

Fig.: Exercise: Approximation of the derivative in Excel

Finite Differences Method (FDM): Order of Accuracy

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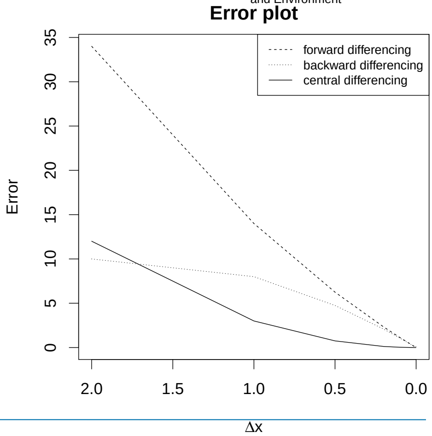
Forward and backward differencing:

- linear relation between Δx and error
- 1st order accuracy

central differencing:

- quadratic relation between Δx and error
- 2nd order accuracy



Leveling of two tanks: Problem statement

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- coefficient of resistance: $\xi = 2{,}0$
- inertia neglected

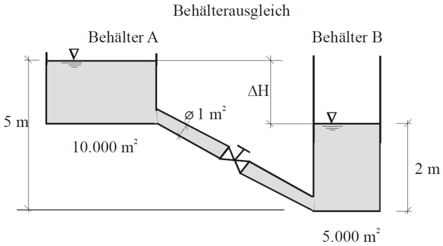

Fig.: Leveling of two tanks connected through a pipe

Leveling of two tanks

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energy conservation:

$$z = \frac{v^2}{2g} \cdot \xi = \frac{Q^2}{A^2 \cdot 2g} \cdot \xi$$

$$\Rightarrow Q = A \cdot \sqrt{\frac{2g \cdot z}{\xi}} \qquad (9)$$

mass conservation:

$$\Delta V = \int Q \, \mathrm{d}t = \int A \cdot \sqrt{\frac{2g \cdot z}{\xi}} \, \mathrm{d}t \qquad (10)$$

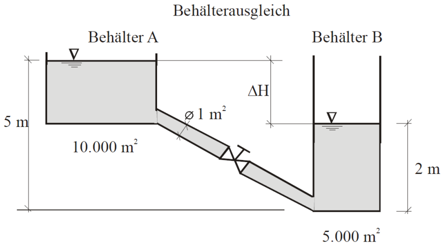

Fig.: Leveling of two tanks connected through a pipe

Leveling of two tanks: Analytical solution

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Change of Water Surface elevation in tank A $\mathrm{d}x$ and tank B $\mathrm{d}y$:

$$\mathrm{d}x = \frac{\mathrm{d}V}{A_\mathrm{A}} = \frac{Q}{A_\mathrm{A}}\ \mathrm{d}t \qquad (11)$$

$$\mathrm{d}y = \frac{\mathrm{d}V}{A_\mathrm{B}} = \frac{Q}{A_\mathrm{B}}\ \mathrm{d}t \qquad (12)$$

$$z = h_\mathrm{A} - h_\mathrm{B} - x - y$$

$$\Rightarrow \mathrm{d}z = -\left(\mathrm{d}x + \mathrm{d}y\right)\ \mathrm{d}t \qquad (13)$$

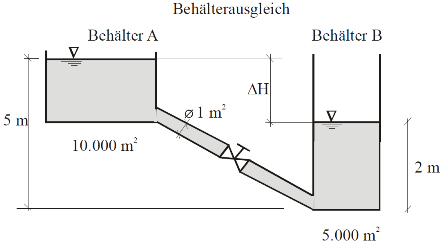

Fig.: Leveling of two tanks connected through a pipe

Leveling of two tanks: Analytical solution

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$\mathrm{d}z$ from equation 13 using equation 11 and 12:

$$\mathrm{d}z = -(\mathrm{d}x + \mathrm{d}y) = -Q \cdot \left(\frac{1}{A_\mathrm{A}} + \frac{1}{A_\mathrm{B}}\right) \mathrm{d}t = -\underbrace{A\sqrt{\frac{2g}{\xi}} \cdot \frac{A_\mathrm{B} + A_\mathrm{A}}{A_\mathrm{A} A_\mathrm{B}}}_{K} \sqrt{z}\, \mathrm{d}t \quad (14)$$

$$\mathrm{d}t = -\frac{1}{K}\frac{\mathrm{d}z}{\sqrt{z}} \quad (15)$$

integration of equation 15:

$$t(z) = -\frac{1}{K}\int_{\Delta H}^{z} \frac{1}{\sqrt{z}}\, \mathrm{d}z = -\frac{1}{K} \cdot 2\sqrt{z}\,\Big|_{\Delta H}^{z} = 2 \cdot \frac{1}{K} \cdot \left(\sqrt{\Delta H} - \sqrt{z}\right) \quad (16)$$

Leveling of two tanks: Analytical solution

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Time until water surface elevations in tanks A and B are fully leveled out ($z = 0$):

$$
\begin{aligned}
t(0) &= 2 \cdot \frac{1}{K} \cdot \left(\sqrt{\Delta H} - \sqrt{z}\right) \\
&= 2 \cdot \frac{1}{A\sqrt{\frac{2g}{\xi} \cdot \frac{A_B + A_A}{A_A A_B}}} \cdot \left(\sqrt{\Delta H} - 0\right) \\
&= \underline{\underline{3\,687\,\mathrm{s}}}
\end{aligned}
$$

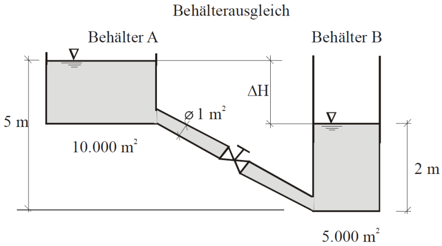

Fig.: Leveling of two tanks connected through a pipe

Leveling of two tanks: Numerical solution

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conservation of mass:

$$\Delta V = Q \cdot \Delta t = A_{\mathrm{A}} \cdot \Delta h_{\mathrm{A}} = A_{\mathrm{B}} \cdot \Delta h_{\mathrm{B}} \quad (17)$$

$$\Rightarrow \Delta h_{\mathrm{A}} = \frac{Q \cdot \Delta t}{A_{\mathrm{A}}} \quad (18)$$

$$\Rightarrow \Delta h_{\mathrm{B}} = \frac{Q \cdot \Delta t}{A_{\mathrm{B}}} \quad (19)$$

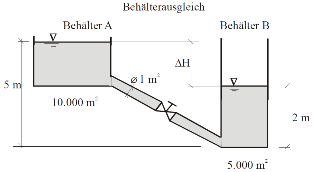

Fig.: Leveling of two tanks connected through a pipe

$$\Delta h_{\mathrm{A}} = h_{\mathrm{A}}^{n} - h_{\mathrm{A}}^{n+1} \quad \Rightarrow h_{\mathrm{A}}^{n+1} = h_{\mathrm{A}}^{n} - \Delta h_{\mathrm{A}} \quad (20)$$

$$\Delta h_{\mathrm{B}} = h_{\mathrm{B}}^{n+1} - h_{\mathrm{B}}^{n} \quad \Rightarrow h_{\mathrm{B}}^{n+1} = h_{\mathrm{B}}^{n} + \Delta h_{\mathrm{B}} \quad (21)$$

Leveling of two tanks: Numerical solution

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Equations 20 and 21 using equations 18 and 19:

$$h_{\mathrm{A}}^{n+1} = h_{\mathrm{A}}^{n} - \frac{\Delta V}{A_{\mathrm{A}}} = h_{\mathrm{A}}^{n} - \frac{Q \cdot \Delta t}{A_{\mathrm{A}}} \tag{22}$$

$$h_{\mathrm{B}}^{n+1} = h_{\mathrm{B}}^{n} + \frac{\Delta V}{A_{\mathrm{B}}} = h_{\mathrm{B}}^{n} + \frac{Q \cdot \Delta t}{A_{\mathrm{B}}} \tag{23}$$

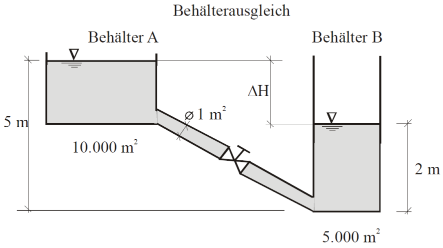

Fig.: Leveling of two tanks connected through a pipe

Leveling of two tanks: Numerical solution

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Q from energy conservation (equation 9)

$$Q = A \cdot \sqrt{\frac{2g \cdot z}{\xi}}$$

different methods to approximate z:

- backward differencing
- central differencing

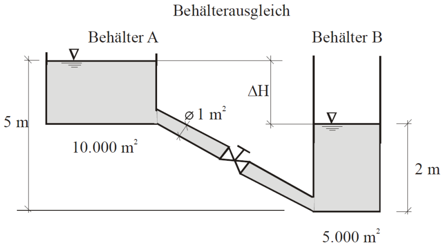

Fig.: Leveling of two tanks connected through a pipe

Leveling of two tanks: Numerical solution

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- backward differencing (Euler method):

$$z = h_{\mathrm{A}}^{n} - h_{\mathrm{B}}^{n} \tag{24}$$

- Zentrale Differenzen (Predictor-Correktor or Runge-Kutta method):

$$\Delta z^{n+\frac{1}{2}} = h_{\mathrm{A}}^{n+\frac{1}{2}} - h_{\mathrm{B}}^{n+\frac{1}{2}} \tag{25}$$

$$\mathrm{mit}: \; h_{\mathrm{A}}^{n+\frac{1}{2}} = \frac{h_{\mathrm{A}}^{n} + h_{\mathrm{A}}^{n+1\star}}{2} \qquad h_{\mathrm{B}}^{n+\frac{1}{2}} = \frac{h_{\mathrm{B}}^{n} + h_{\mathrm{B}}^{n+1\star}}{2} \tag{26}$$

with $h_{\mathrm{A}}^{n+1\star}$ and $h_{\mathrm{B}}^{n+1\star}$ estimated using backward differencing.

Leveling of two tanks: Numerical solution

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Numerical solution in Excel applying the Euler method:

	A	B	C
1	g	m/s²	9.81
2	A	m²	1
3	AA	m²	10000
4	AB	m²	5000
5	zeta	1	2
6	Delta t	s	100

Fig.: Leveling of two tanks input

	A	B	C	D	E	F	G	H	I	J
1	n	t	hA	hB	z	Q	Delta V	hA_(n+1)	hB_(n+1)	t analytisch
2		s	m	m	m	m³/s	m³	m	m	s
3	0	0	5.000	2.000	3.000	5.425	542.494	4.946	2.108	0
4	1	100	4.946	2.108	2.837	5.276	527.574	4.893	2.214	101
5	2	200	4.893	2.214	2.679	5.126	512.648	4.842	2.317	203
6	3	300	4.842	2.317	2.525	4.977	497.715	4.792	2.416	304
7	4	400	4.792	2.416	2.376	4.828	482.776	4.744	2.513	406
8	5	500	4.744	2.513	2.231	4.678	467.830	4.697	2.606	507
9	6	600	4.697	2.606	2.091	4.529	452.876	4.652	2.697	609
10	7	700	4.652	2.697	1.955	4.379	437.914	4.608	2.784	711
11	8	800	4.608	2.784	1.823	4.229	422.943	4.566	2.869	812
12	9	900	4.566	2.869	1.697	4.080	407.963	4.525	2.951	914
13	10	1000	4.525	2.951	1.574	3.930	392.972	4.485	3.029	1016
14	11	1100	4.485	3.029	1.456	3.780	377.971	4.448	3.105	1118
15	12	1200	4.448	3.105	1.343	3.630	362.958	4.411	3.177	1220
16	13	1300	4.411	3.177	1.234	3.479	347.932	4.377	3.247	1322
17	14	1400	4.377	3.247	1.130	3.329	332.892	4.343	3.313	1424
18	15	1500	4.343	3.313	1.030	3.178	317.836	4.311	3.377	1527
19	16	1600	4.311	3.377	0.934	3.028	302.764	4.281	3.438	1629
20	17	1700	4.281	3.438	0.844	2.877	287.673	4.252	3.495	1732
21	18	1800	4.252	3.495	0.757	2.726	272.561	4.225	3.550	1834

Fig.: Numerical solution in Excel applying the Euler method

Leveling of two tanks: Numerical solution

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Numerical solution in Excel applying the Predictor-Corrector method:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	n	t	hA	hB	z	Q	Delta V	hA_(n+1)*	hB_(n+1)*	t analytisch	hA_(n+0.5)	hB_(n+0.5)	z Präd.-Korr.	Q	Delta V	hA_(n+1)	hB_(n+1)
2		s	m	m	m	m^3/s	m^3	m	m	s	m	m	m	m^3/s	m^3	m	m
3	0	0	5.000	2.000	3.000	5.425	542.494	4.946	2.108	0	4.973	2.054	2.919	5.351	535.086	4.946	2.107
4	1	100	4.946	2.107	2.839	5.278	527.781	4.894	2.213	100	4.920	2.160	2.760	5.204	520.371	4.894	2.211
5	2	200	4.894	2.211	2.683	5.131	513.067	4.843	2.314	200	4.869	2.262	2.606	5.057	505.656	4.844	2.312
6	3	300	4.844	2.312	2.532	4.984	498.354	4.794	2.412	300	4.819	2.362	2.457	4.909	490.941	4.795	2.410
7	4	400	4.795	2.410	2.384	4.836	483.640	4.746	2.507	400	4.771	2.459	2.312	4.762	476.226	4.747	2.506
8	5	500	4.747	2.506	2.242	4.689	468.927	4.700	2.599	500	4.724	2.553	2.171	4.615	461.511	4.701	2.598
9	6	600	4.701	2.598	2.103	4.542	454.214	4.656	2.689	600	4.678	2.643	2.035	4.468	446.796	4.656	2.687
10	7	700	4.656	2.687	1.969	4.395	439.501	4.612	2.775	700	4.634	2.731	1.903	4.321	432.081	4.613	2.774
11	8	800	4.613	2.774	1.839	4.248	424.788	4.571	2.859	800	4.592	2.816	1.776	4.174	417.366	4.571	2.857
12	9	900	4.571	2.857	1.714	4.101	410.076	4.530	2.939	900	4.551	2.898	1.653	4.027	402.651	4.531	2.938
13	10	1000	4.531	2.938	1.593	3.954	395.363	4.492	3.017	1000	4.511	2.977	1.534	3.879	387.936	4.492	3.015
14	11	1100	4.492	3.015	1.477	3.807	380.651	4.454	3.091	1100	4.473	3.053	1.420	3.732	373.221	4.455	3.090
15	12	1200	4.455	3.090	1.365	3.659	365.939	4.418	3.163	1200	4.437	3.127	1.310	3.585	358.506	4.419	3.162
16	13	1300	4.419	3.162	1.257	3.512	351.227	4.384	3.232	1300	4.402	3.197	1.205	3.438	343.791	4.385	3.230
17	14	1400	4.385	3.230	1.154	3.365	336.515	4.351	3.298	1400	4.368	3.264	1.104	3.291	329.076	4.352	3.296
18	15	1500	4.352	3.296	1.056	3.218	321.804	4.320	3.361	1500	4.336	3.328	1.007	3.144	314.360	4.320	3.359
19	16	1600	4.320	3.359	0.961	3.071	307.093	4.290	3.421	1600	4.305	3.390	0.915	2.996	299.645	4.290	3.419
20	17	1700	4.290	3.419	0.871	2.924	292.383	4.261	3.478	1700	4.276	3.448	0.828	2.849	284.930	4.262	3.476
21	18	1800	4.262	3.476	0.786	2.777	277.673	4.234	3.532	1800	4.248	3.504	0.744	2.702	270.215	4.235	3.530

Fig.: Numerical solution in Excel applying the Predictor-Corrector method

Leveling of two tanks: Numerical solution

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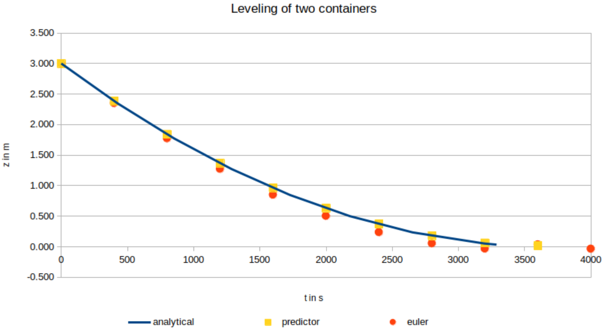

Fig.: Levelling of two containers: comparison of solutions

Leveling of two tanks: Numerical solution

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Solution using the prepared Excel-Worksheet by Maurer (2010)
`Behaelterausgleich_Maurer.xlsm`
note: access VBA-code via `Developer Tools > VBA`
tasks:

- Which convergence criteria is used?
- How long does it take for the water surfaces in the tanks to level out?
- How long does it take for the water surfaces in the tanks to level out when the pipe diameter is doubled?
- How can the difference between the Euler and the Predictor-Corrector method be explained?

Leveling of two tanks: Summary

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Tab.: Comparison between an analytical and a numerical solution

analytical solution	**numerical solution**
available in special cases only	can be derived for any problem
no discretization necessary	suitable discretization necessary for accurate results
function can be evaluated at individual points	function needs to be solved for the entire domain

Mass transport: The Sandoz/Rhine accident 1986

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Fig.: Plainly Difficult: “A Brief History of: The Sandoz Chemical Disaster (Short Documentary)” `https://www.youtube.com/watch?v=6RjTvN2QhSY` (7:08 min)

Mass transport: The Sandoz/Rhine accident 1986

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- one of the severest environmental disasters caused by humans in Europe
- chemically polluted water from discharged into the Rhine river
- Rhine polluted over a length of 400 km
- initiation of new developments in water protection
- amongst others development of the Rhine-Alarm model (Mazijk et al., 1991)

(IKSR, 2016)

Fig.: source: Badische Zeitung/dapd (IKSR, 2016)

Mass transport: The Sandoz/Rhine accident 1986

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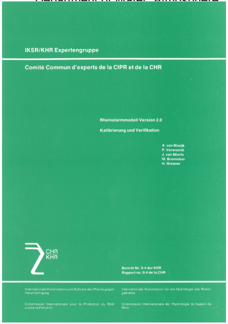
Aim of the model is to predict

- time of arrival of the polluted water
- maximum concentration of pollutants
- end of contamination

on any point along the river length (Mazijk et al., 1991).
allows to warn e. g.:

- waterworks
- fishers

`https://www.chr-khr.org/de/veroffentlichung/rheinalarmmodell-version-20-kalibrierung-und-verifikation`



IKSR/KHR Expertengruppe

Comité Commun d'experts de la CIPR et de la CHR

Rheinalarmmodell Version 2.0

Kalibrierung und Verifikation

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CHR
KHR

Bericht Nr. II-4 der KHR
Rapport no. II-4 de la CHR

Internationale Kommission zum Schutze des Rheins gegen Verunreinigung

Internationale Kommission für die Hydrologie des Rheingebietes

Commission Internationale pour la Protection du Rhin contre la Pollution

Commission Internationale de l'Hydrologie du bassin du Rhin

Fig.: Validation report of the Rhine-Alarm model (Mazijk et al., 1991)

Mass transport: Governing equations

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- Advection equation:

$$\frac{\partial c}{\partial t} + \boldsymbol{u} \cdot \nabla c = 0 \tag{27}$$

$$\frac{\partial c}{\partial t} + u \cdot \frac{\partial c}{\partial x} = 0 \tag{28}$$

- Advection-Diffusion equation:

$$\frac{\partial c}{\partial t} + u \cdot \frac{\partial c}{\partial x} + D_0 \frac{\partial^2 c}{\partial x^2} = 0 \tag{29}$$

for comparison: diffusion only

$$\frac{\partial \phi}{\partial t} + \Gamma \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{30}$$

Mass transport: Governing equations

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Examples:

- advection only: transport of a solid
- diffusion only: heat transport in a solid
- advection and diffusion: dissolved substance in a fluid

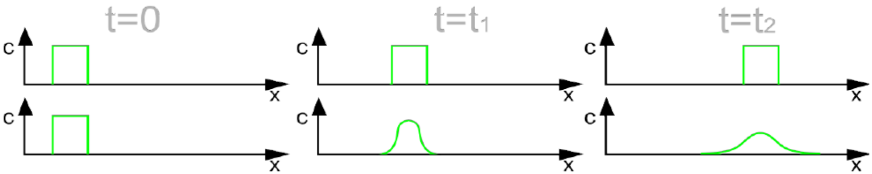

Fig.: Scheme of mass transport without diffusion (top) and with diffusion (bottom) (Maurer, 2010)

Mass transport: Governing equations

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Semi-empirical calculation of the coefficient of dispersion D_0:

$$D_0 = \alpha \cdot \frac{u^2 \cdot B_S^2}{\bar{h} \cdot u_\star} \qquad \text{mit } u_\star = \frac{u \cdot \sqrt{g}}{C} = \frac{u \cdot g^{\frac{1}{2}}}{k_{\mathrm{St}} \cdot \sqrt[3]{\bar{h}}} \tag{31}$$

Dispersion has the same effect as diffusion, but the coefficient of dispersion can be several orders of magnitudes higher than the coefficient of diffusion.

α .... dimensionless constant $[\alpha] = 1$
B_S .... effective river width $[B_S] = \mathrm{m}$
$\bar{h}$ .... average water depth $[\bar{h}] = \mathrm{m}$
$u_\star$ .... shear stress velocity $[u_\star] = \mathrm{m\,s^{-1}}$
g .... gravitational acceleration $[g] = \mathrm{m\,s^{-2}}$
C .... Chézy-Coefficient $[C] = \mathrm{m^{1/2}\,s^{-1}}$
k_{St} .... Strickler-Coefficient $[k_{\mathrm{St}}] = \mathrm{m^{1/3}\,s^{-1}}$

Mass transport: Governing equations

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stagnant water zones: extension for non-uniform velocity distributions along the river width (Chatwin, 1980)

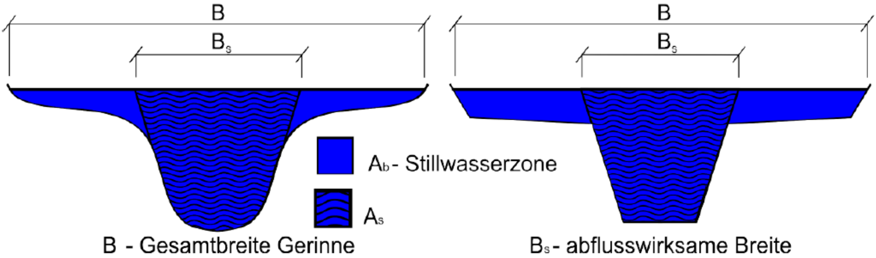

Fig.: Scheme of stagnant water zones (Maurer, 2010)

Mass transport: Governing equations

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Transport velocity c_v:

$$c_v = \frac{u}{1+\beta} \qquad \text{mit } \beta = \kappa \cdot \frac{A_b}{A_s} \tag{32}$$

c_v .... mass transport velocity $[c_v] = \mathrm{m\,s^{-1}}$
β .... portion of stagnant water zones $[\beta] = 1$
κ .... exchange coefficient $[\kappa] = 1$
A_b .... storing cross sectional area $[A_b] = \mathrm{m^2}$
A_s .... effective cross sectional area $[A_s] = \mathrm{m^2}$

ad κ (exchange between stagnant and effective water zones): set $\kappa = 1$ for complete exchange

Mass transport: Boundary and initial conditions

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Boundary and initial conditions:

- total mass added at one particular instance in time and at one particular point along the river
- steady-state and uniform flow[1]
- initially no-substance in the river

[1] extension for non-uniform flow already implemented in the Rhine-Alarm model (Mazijk et al., 1991)

Mass transport: Analytical solution

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Equation 29 using boundary and initial conditions from above can be solved analytically using Taylor-polynomials (Fischer et al., 1979).

$$c(x,t) = \frac{\frac{M}{Q}}{\sqrt{4\pi D_0 \frac{t}{u_s^2}}} \cdot \exp\left(-\frac{\left(t - \frac{x}{u_s}\right)^2}{4 D_0 \frac{t}{u_s^2}}\right) \tag{33}$$

Mass transport: Analytical solution

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Taking stagnant water zones into account results in the following equation:

$$c(x,t) = \frac{\frac{M}{Q}}{\sqrt{4\pi D_0 \frac{t}{c_v^2}}} \cdot \exp\left(-\frac{\left(t-\frac{x}{c_v}\right)^2}{4 D_0 \frac{t}{c_v^2}}\right) \cdot \left(1 + \frac{G_t}{6} \cdot H_3 \cdot \frac{t-\frac{x}{c_v}}{\sqrt{2 D_0 \frac{t}{c_v^2}}}\right) \quad (34)$$

$H_3[z] = z^3 - 3z$ Hermitian polynomial ($z = \frac{t-\mu_t}{\sigma_t}$)

μ_t .... mean $[\mu_t] = \mathrm{s}$

σ_t .... variance $[\sigma_t] = \mathrm{s}^2$

$G_t = \frac{|g_t|}{|\sigma_t^3|}$ .... skewness $[G_t] = 1$

g_t .... 3. moment $[g_t] = \mathrm{s}^3$

Mass transport: Analytical solution

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Solution using the prepared Excel-Worksheet by Maurer (2010)
`masstransport.xlsm`
note: access VBA-code via `Developer Tools > VBA`
Input values river:

- length of the river section: $L = 80\,\mathrm{km}$
- Mannings' coeffecient: $\frac{1}{n} = k_{\mathrm{St}} = 35\,\mathrm{m}^{1/3}\,\mathrm{s}^{-1}$
- discharge $Q = 1\,800\,\mathrm{m}^3\,\mathrm{s}^{-1}$
- effective cross sectional area: $A_s = 750\,\mathrm{m}^2$
- effective river width: $B_s = 75\,\mathrm{m}$
- storing cross sectional area: $A_D = 120\,\mathrm{m}^2$
- water depth: $h = 5\,\mathrm{m}$

Mass transport: Analytical solution

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Input values substance:

- mass of added substance: $m = 2\,\mathrm{kg}$
- coefficient $\kappa = 1$
- coefficient of dispersion: $\alpha = 1$
- skewness: $G_t = 1$

Tasks `masstransport.xlsm`

- When is the maximum concentration occuring in the middle of the river section?
- How high is the maximum concentration in the middle of the river section
- What would be the respective values when stagnant water zones are neglected?

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Department of Water, Atmosphere and Environment

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